Automatic Control

Chapter two

Mathematical modeling of control systems

By

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A u t o m a t i c C o n t r o l

Mathematical modeling of control systems

Mathematical modeling

The dynamics of many systems, whether they are mechanical, electrical, thermal, economic, biological, and so on, may be described in terms of differential equations

Linear Systems. A system is called linear if the principle of superposition applies. The principle of superposition states that the response produced by the simultaneous application of two different forcing functions is the sum of the two individual responses

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TRANSFER FUNCTION

Transfer Function. The transfer function of a linear, time-invariant, differential equation system is defined as the ratio of the Laplace transform of the output (response function) to the Laplace transform of the input (driving function) under the assumption that all initial conditions are zero.

Consider the linear time-invariant system defined by the following differential equation:

$$a_0 \overset{(n)}{y} + a_1 \overset{(n-1)}{y} + \cdots + a_{n-1}\dot{y} + a_n y = b_0 \overset{(m)}{x} + b_1 \overset{(m-1)}{x} + \cdots + b_{m-1}\dot{x} + b_m x \qquad (n \geq m)$$

$$\text{Transfer function} = G(s) = \left.\frac{\mathscr{L}[\text{output}]}{\mathscr{L}[\text{input}]}\right|_{\text{zero initial conditions}}$$

$$= \frac{Y(s)}{X(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1}s + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$$

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Comments on Transfer Function

1. The transfer function of a system is a mathematical model in that it is an operational method of expressing the differential equation that relates the output variable to the input variable.
2. The transfer function is a property of a system itself, independent of the magnitude and nature of the input or driving function.
3. The transfer function includes the units necessary to relate the input to the output; however, it does not provide any information concerning the physical structure of the system. (The transfer functions of many physically different systems can be identical.)

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Comments on Transfer Function

4. If the transfer function of a system is known, the output or response can be studied for various forms of inputs with a view toward understanding the nature of the system.

5. If the transfer function of a system is unknown, it may be established experimentally by introducing known inputs and studying the output of the system. Once established, a transfer function gives a full description of the dynamic characteristics of the system, as distinct from its physical description

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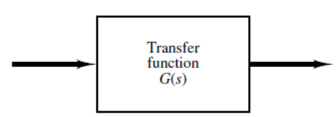
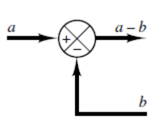
Block diagram

Block Diagram:- is a pictorial representation of the system governing equations. In general, it is a box with two arrows. One is the input and the other is the output.

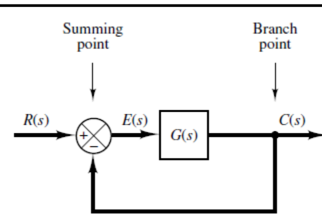


Block Diagram element

Summing Point.



Block Diagram of a Closed-Loop System



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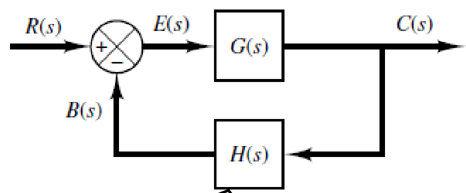

Used to convert the form of the output signal to that of the input signal

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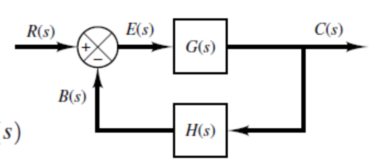
Open-Loop Transfer Function and Feed forward Transfer Function



$$\text{Open-loop transfer function} = \frac{B(s)}{E(s)} = G(s)H(s)$$

$$\text{Feedforward transfer function} = \frac{C(s)}{E(s)} = G(s)$$

Closed-Loop Transfer Function

$$C(s) = G(s)E(s)$$
$$E(s) = R(s) - B(s)$$
$$= R(s) - H(s)C(s)$$

$$C(s) = G(s)[R(s) - H(s)C(s)]$$

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

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Drawing block diagram

Example :- draw a single block diagram that represents the following system

$$U(s) = G_f R(s) + G_c E(s)$$

$$C(s) = G_p\left[D(s) + G_1 U(s)\right]$$

$$E(s) = R(s) - HC(s)$$

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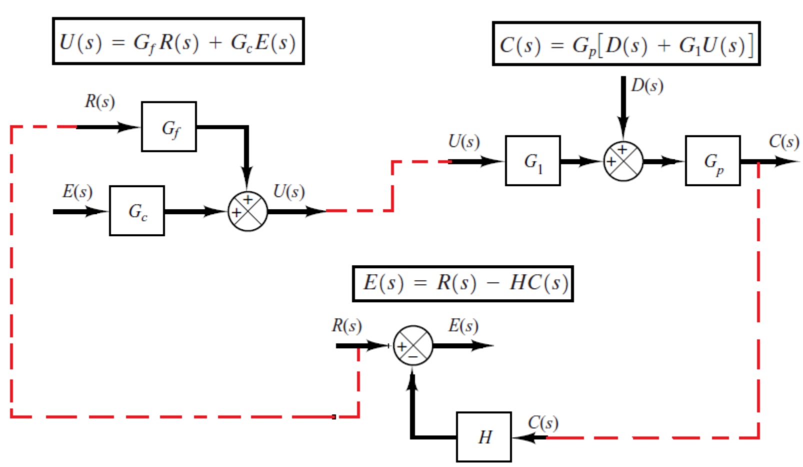
Drawing block diagram



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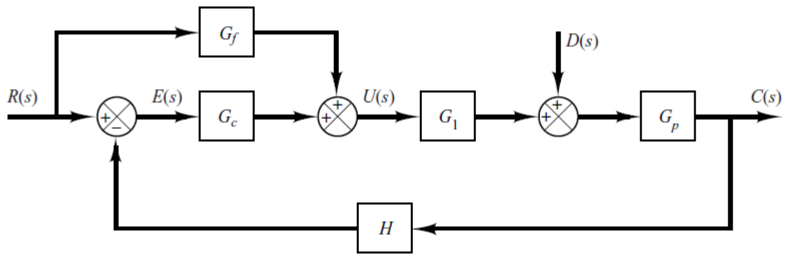
Drawing block diagram



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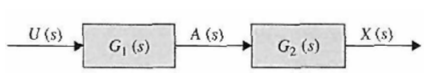
Block diagram reduction

Consecutive blocks



$$\begin{aligned} A(s) &= G_1(s)U(s) \\ X(s) &= G_2(s)A(s) \end{aligned} \Rightarrow X(s) = G_1(s)G_2(s)U(s)$$

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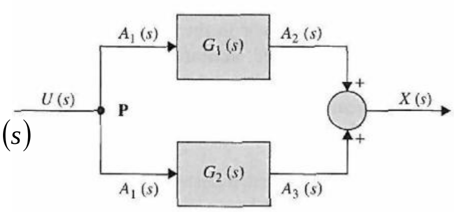
Block diagram reduction

Nodes and comparators

$$A_1(s) = U(s)$$
$$A_2(s) = G_1(s)A_1(s) = G_1(s)U(s)$$
$$A_3(s) = G_2(s)A_1(s) = G_2(s)U(s)$$
$$X(s) = A_2 + A_3 = G_1(s)U(s) + G_2(s)U(s)$$
$$\Rightarrow X(s) = U(s)[G_1(s) + G_2(s)]$$
$$\Rightarrow G(s) = G_1(s) + G_2(s)$$



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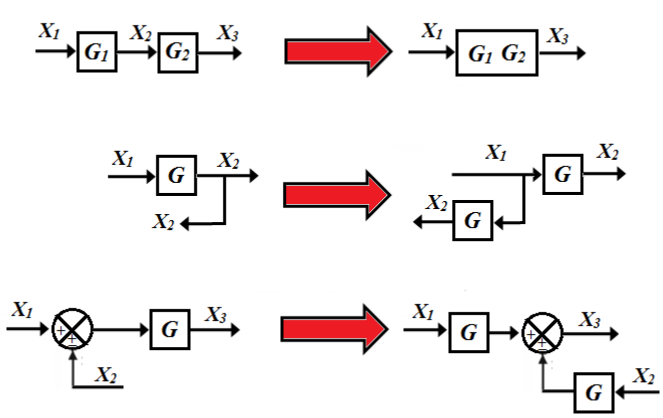
Block Diagram Reduction

➢It is preferred in many cases to reduce the complex block diagram to a single block diagram relates the excitation with the response



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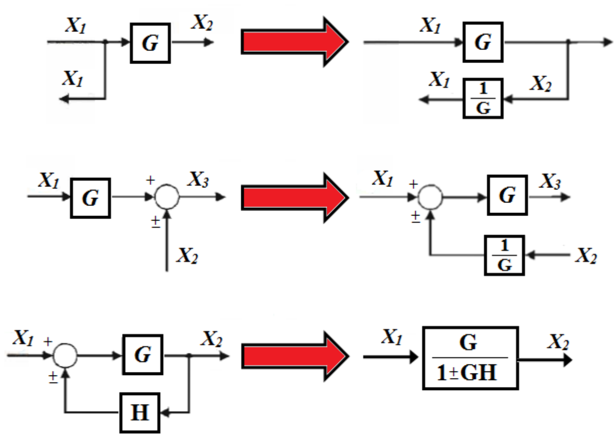
Block Diagram Reduction



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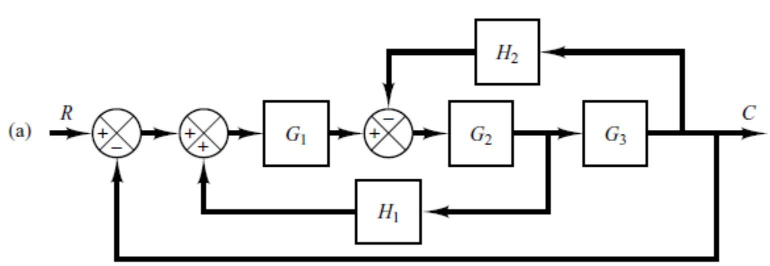
Block Diagram Reduction

EXAMPLE 2–1

Consider the system shown in Figure 2–13(a). Simplify this diagram.



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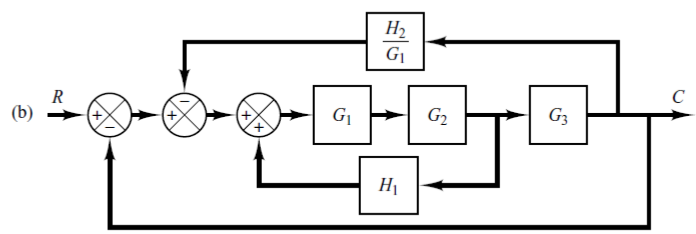
Block Diagram Reduction

EXAMPLE 2–1

Solution

By moving the summing point of the negative feedback loop containing *H2 outside the positive* feedback loop containing *H1, we obtain Figure 2–13(b).*



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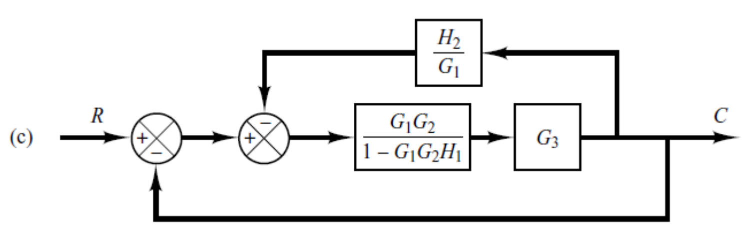
Block Diagram Reduction

EXAMPLE 2–1

Solution

Eliminating the positive feedback loop, we have Figure 2–13(c).



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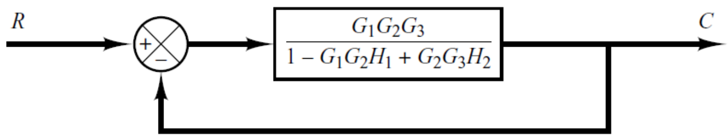
Block Diagram Reduction

EXAMPLE 2–1

Solution

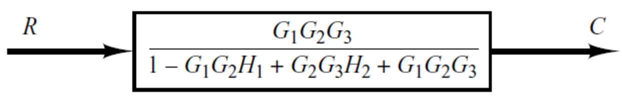
The elimination of the loop containing *H2/G1 gives Figure 2–13(d).*

(d)



Finally, eliminating the feedback loop results in Figure 2–13(e).

(e)



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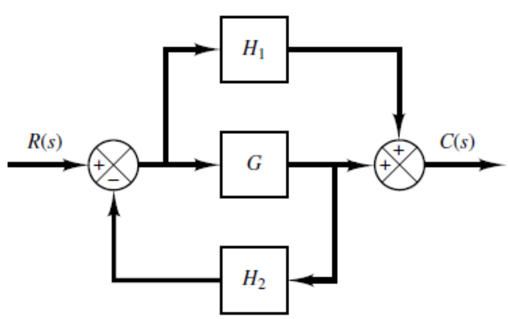
Block Diagram Reduction

A–2–1. Simplify the block diagram shown in Figure



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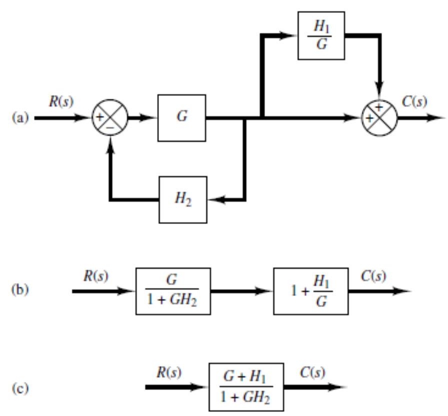
Block Diagram Reduction

EXAMPLE A–2–1.

Solution



1. First, move the branch point of the path involving *H1 outside the loop involving H2* as shown in Figure (a)
2. Then eliminating two loops results in Figure (b).
3. Combining two blocks into one gives Figure (c).

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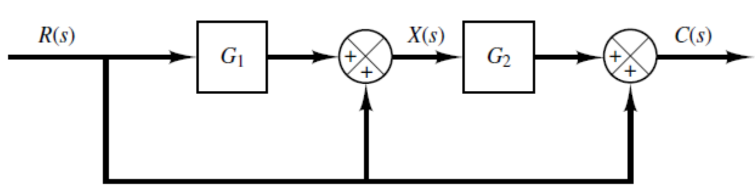
Block Diagram Reduction

A–2–2. Simplify the block diagram shown in Figure 2–19. Obtain the transfer function relating C(s) and R(s).



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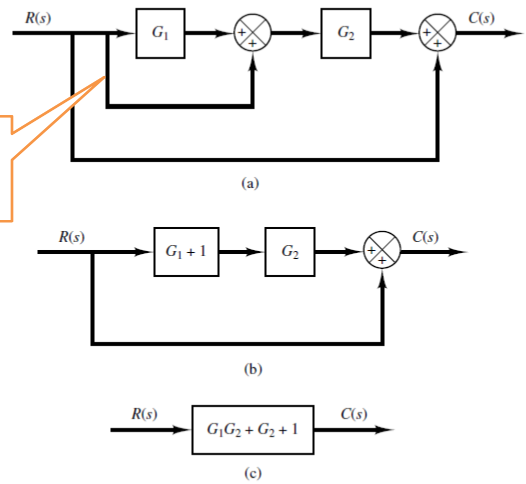
Block Diagram Reduction

A–2–2.

Solution

Redrawn for more identification



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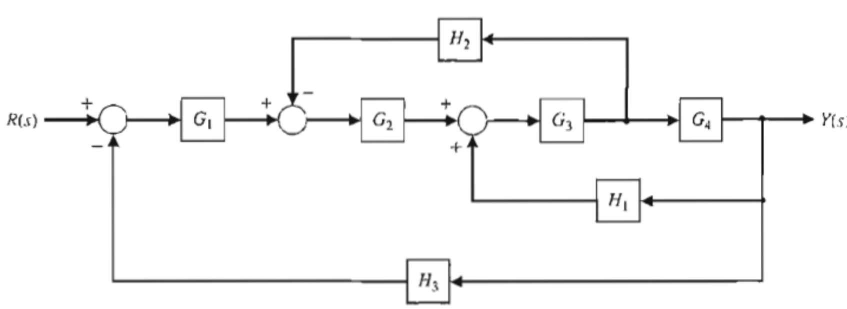
Block Diagram Reduction

Example 2.7 Dorf (2008)

Reduce the following block diagram to single block diagram system



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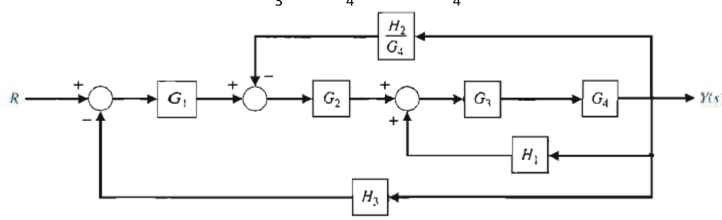
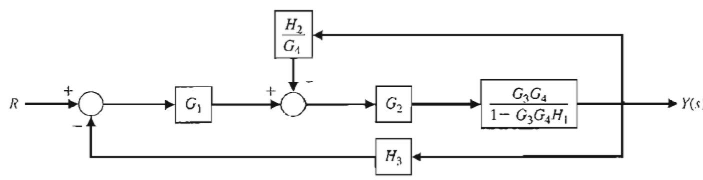
Solution

1. Move the node between G_3 and G_4 to after G_4:



2. Reduce the feedback system G_3, G_4 and H_1:



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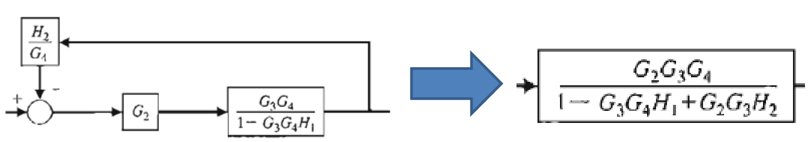
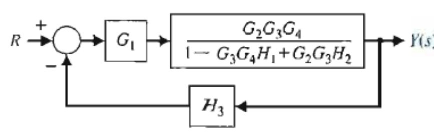
Example [1]

Solution

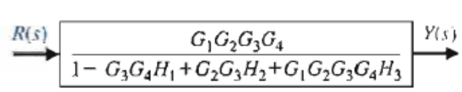
3. Reduce the feedback system



The system become



Reduce the final feed back system



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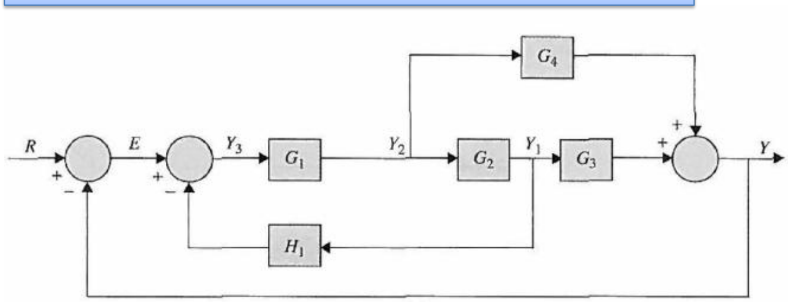
Block Diagram Reduction

EXAMPLE 3-1-5 Golnaraghi (2010)

Reduce the following block diagram to single block diagram system



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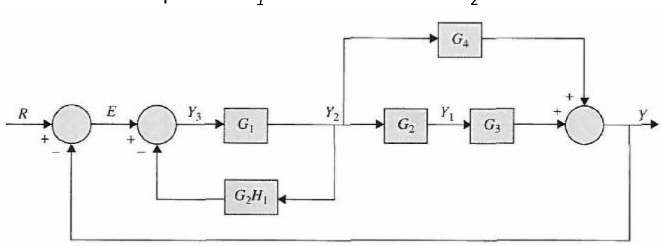
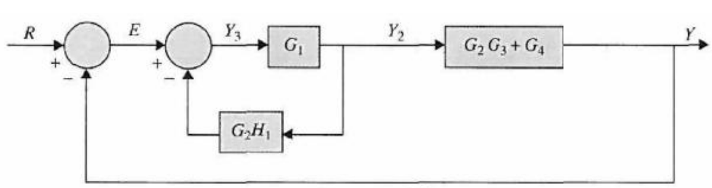
Solution

1. move the branch point at Y_1 to the left of block G_2



2. combining the blocks G_2, G_3, and G_4



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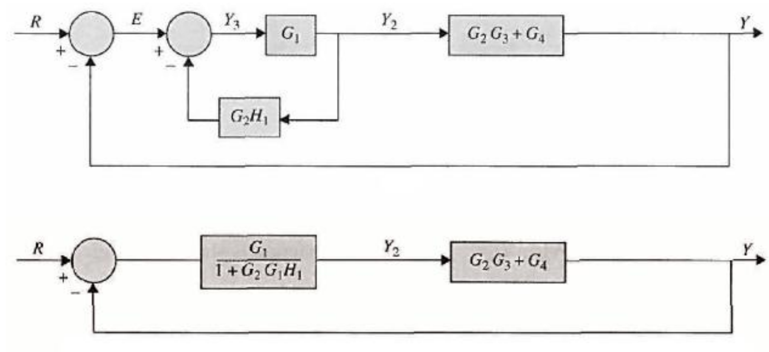
Solution

3. eliminating the two feedback loops



$$\frac{Y(s)}{E(s)} = \frac{G_1G_2G_3 + G_1G_4}{1 + G_2G_3H_1 + G_1G_2G_3 + G_1G_4}$$

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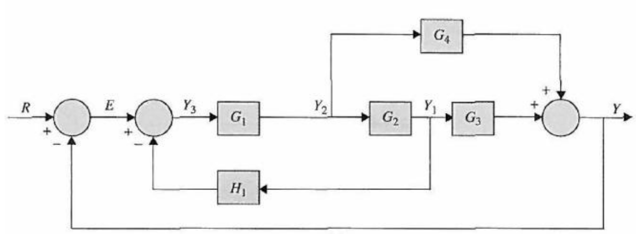
Obtaining system functions from block diagram

Consider the same system from example **3-1-5** Golnaraghi (2010), find the system equations in s-domain



Solution

$$E = R - Y$$

$$Y_3 = E - (H_1)Y_1$$

$$Y_2 = (G_1)Y_3$$

$$Y_1 = (G_2)Y_2$$

$$Y = (G_3) + (G_4)Y_2$$